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On the everywhere divergence of Vilenkin—Fourier series

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

In this paper we are concerned with everywhere divergence of Vilenkin—Fourier series. The Vilenkin systems are generalizations of the Walsh system. It is well known that A. N. Kolmogoroff gave the first example for integrable function with everywhere divergent Fourier series. The corresponding question for Wals system wash solved by F. SCHIPP [4], [5], and the construction for arbitrary Vilenkin system due to P. SIMON [7]. There are many interesting new results in this theme. For example S. SH. GALSTIAN [2] proved the existence of an integrable function with everywhere divergent Fourier series, the Fourier coefficients of which tend to zero so rapid as possible. It is natural to ask whether the analogue theorem is true for Vilenkin systems or not. We show that a similar theorem is valid for the so called bounded Vilenkin systems. Our theorem is sharper than the theorem concerning the trigonometric system [2], because we construct an appropriate function, which is not only integrable, but belongs to a function class connected with a Hardy type space, too. In our construction the Vilenkin polynomials introduced by P. Simon [6], [7] play important role. We give proof only for the case of Vilenkin groups, but this proof can be easily transferred for the case of $[0, 1)$.

2. Preliminaries

Let $m := (m_k, k \in \mathrm{N})$ $(\mathrm{N}=0, 1, \ldots)$ be a sequence of natural numbers, the terms of which are not less than 2. Denote by Z_{m_k} $(k \in \mathrm{N})$ the discrete cyclic group of order m_k. We define the so called Vilenkin group G_m as the direct product of Z_{m_k}'s $(k \in \mathrm{N})$. Thus G_m is a compact Abelian group, the elements of which are represented in the

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form $x=(x_0, x_1, \ldots, x_k, \ldots)$ $(0 \leqq x_k < m_k, x_k, k \in \mathbf{N})$. μ denotes the normalized Haar measure on G_m.

Introduce the next notations:

$$M_0 := 1, \quad M_{k+1} := \prod_{i=1}^{k} m_i \quad (k \in \mathbf{N}).$$

It is clear that every $n \in \mathbf{N}$ can be uniquely rewritten in the form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (0 \leqq n_k < m_k, \quad n_k \in \mathbf{N}).$$

We shall need the following subsets of G_m:

$$I_n(x) := \{y \in G_m | y_k = x_k, \quad k < n\} \quad (n \in \mathbf{N}, \quad x \in G_m).$$

Obviously $\mu(I_n(x)) = M_n^{-1}$. Let $\hat{G}_m := \{\psi_n, n \in \mathbf{N}\}$ the character system of G_m ordered in the Walsh—Paley sense, i.e.

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k},$$

where

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (k \in \mathbf{N}, x \in G_m, i := \sqrt{-1}).$$

It is known that $\hat{G}_m$ is a complete orthonormal system with respect to μ. The Vilenkin system is said to be bounded if $\lim \sup m < \infty$.

Denote $L(G_m)$ the space of μ-integrable functions and define the norm of $f \in L(G_m)$ as $\|f\|_1 := \int_{G_m} |f| \, d\mu$. If $f \in L(G_m)$ then let

$$\hat{f}(k) := \int_{G_m} f \bar{\psi}_k \, d\mu \quad (k \in \mathbf{N})$$

the k-th Vilenkin—Fourier coefficient of f,

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \boldsymbol{P} := 1, 2, \ldots)$$

he n-th partial sum of Vilenkin—Fourier series of f,

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \boldsymbol{P})$$

the n-th Dirichlet kernel with respect to the Vilenkin system $\hat{G}_m$. Define the functiont $\tau_h f$ $(h \in G_m, f \in L(G_m))$ as follows

$$\tau_h f(x) := f(x \dot{-} h) \quad (x \in G_m),$$

where $\dot{-}$ is the inverse of the group operation which is denoted by $\dot{+}$.

It is known [8] that

$$D_{M_n}(x)=\begin{cases} M_n, & \text{if } x\in I_n(0),\\ 0, & \text{if } x\in G_m\setminus I_n(0),\end{cases} \tag{1}$$

where $\mathbf{0}:=(0,0,\dots)\in G_m$.

We need the following notations:

$$\Delta f:=\{k\in\mathbf{N}\,|\,\hat{f}(k)\neq 0\},$$

$$\Omega f:=\sup_{n,l}|S_nf-S_lf|\quad (f\in L(G_m)).$$

Now we define a Hardy type space $H(G_m)$. Let us denote by f^* the following maximal function of $f\in L(G_m)$

$$f^*:=\sup_n|S_{M_n}f|\quad (n\in\mathbf{N}).$$

We say that $f\in L(G_m)$ belongs to $H(G_m)$ if $f^*\in L(G_m)$, and let $\|f\|_H:=\|f^*\|_1$. $H(G_m)$ is a Banach space with this norm. If Φ: $[0,\infty)\to[0,\infty)$ is an increasing function, then we denote by $H\Phi(H)$ the class of μ-integrable functions for which

$$f^*\Phi\circ f^*\in L(G_m)$$

($\circ$ stands for composition of functions). In this paper we prove the next statement for bounded Vilenkin systems.

Theorem. *Let Φ: $[0,+\infty)\to[0,+\infty)$ be an increasing function with $\Phi(n)=$ $=o(\log\log n)$ $(n\to+\infty)$ and $(\alpha_n, n\in\mathbf{N})$ a decreasing sequence tending to zero for which*

$$\sum_{n=0}^{\infty}\alpha_n^2=\infty.$$

Then there exists a function $f\in H\Phi(H)$ such that

$$|\hat{f}(k)|\leqq\alpha_k\quad (k\in\mathbf{N})$$

and the Vilenkin—Fourier series of f diverges everywhere.

Remark. The Vilenkin systems are orthonormal systems with respect to the Lebesgue measure on $[0,1)$, therefore all the concepts like Vilenkin—Fourier series, maximal function etc. can be introduced also for functions of $L[0,1)$. The above Theorem can be formulated for this case too. It is not hard to check, that all the considerations used in the proof of Theorem can be transferred for this case. This is based on the fact, that there is a close connection between G_m and $[0,1)$, namely

$$\lambda\colon G_m\to[0,1),\quad \lambda(x):=\sum_{k=0}^{\infty}\frac{x_k}{M_{k+1}}$$

is an almost one-one and measure preserving mapping. C will denote an absolute positive, but not always the same constant in this paper.

3. Two lemmas

In order to prove Theorem we need two lemmas. Let us denote by P_n $(n\in\mathbf{P})$ the set of Vilenkin polynomials of order n, i.e.

$$P_n := \{g\in L(G_m)|\sup \Delta g < n\}.$$

Let $\hat{G}_m$ be an arbitrary Vilenkin system. Then the following lemma is true.

Lemma 1. *For all $n, p\in\mathrm{N}$ and $0\leqq j<M_p$ $(j\in\mathrm{N})$ there exists a Vilenkin polynomial $Q_{n,p,j}$ such that*

(i) $Q_{n,p,j}\in P_{M_{M_{n+1}+p}}$,

(ii) $\|Q_{n,p,j}\|_1 = 1$,

(iii) $\Omega Q_{n,p,j}(x) > CM_p n \quad (x\in I_p(e_j))$,

$\left(\text{where } e_j:=(j_0, j_1, \ldots, j_{p-1}, 0, \ldots)\in G_m,\ jM_p^{-1}=\sum_{k=0}^{p-1} j_k M_{k+1}^{-1}\right)$,

(iv) $\operatorname{supp} Q_{n,p,j}\subset I_p(e_j)$.

Proof. Let the numbers n, p, j and the Vilenkin system $\hat{G}_m$ be fixed. Define the sequence $m'=(m_k', k\in\mathrm{N})$ as follows

$$m_k':= m_{k+p} \quad (k\in\mathrm{N}).$$

m' generates the Vilenkin group $G_{m'}$. We shall denote by Q_n the same Vilenkin polynomial as in [7] (pp. 361—362). The corresponding polynomial with respect to $\hat{G}_{m'}$ is denoted by Q_n'. It is shown in [7] that

$$\|Q_n'\|_1 = 1, \quad \Omega Q_n'(x) > Cn, \quad Q_n'\in P_{M'_{M'_{n+1}}}$$

where

$$M_l':= \prod_{i=0}^{l-1} m_i' = \prod_{i=0}^{l-1} m_{p+i}(x\in G_m, l\in\mathrm{N}).$$

By means of Q_n' we introduce the Vilenkin polynomial $Q_{n,p}$ on G_m as follows

$$Q_{n,p}(x):= Q_n'(y), \quad \text{where} \quad y_k = x_{k+p} \quad (x\in G_m, y\in G_m, \quad k\in\mathrm{N}).$$

It is clear from the definition of $Q_{n,p}$, that if $Q_n'=\sum a_i\psi_i'$ (ψ_i' is the i-th element of $\hat{G}_{m'}$), then $Q_{n,p}=\sum a_i\psi_{iM_p}$, and $\|Q_{n,p}\|_1=1$, $\Omega Q_{n,p}(x)>Cn$, $Q_{n,p}\in P_{M_{M_{n+1}+p}}$ $(x\in G_m)$. Define $Q_{n,p,j}:=\tau_{e_j}D_{M_p}Q_{n,p}$. Applying (1) it is easy to check that $Q_{n,p,j}$ has the desired properties. Lemma 1 is proved.

The second lemma is a modification of a lemma of Stečkin [1].

Lemma 2. *If* $(\alpha_n, n\in\mathbf{N})$ *tends monotonely to zero and* $\sum_{n=1}^{\infty}\alpha_n^2=\infty$, *then for all* $s, q\in\mathbf{N}$ *there exist* $t\in\mathbf{N}$ *and* $p_1\leqq p_2\leqq\ldots\leqq p_t$ $(p_i\in\mathbf{N},\ i=1, \ldots, t)$ *such that*

(i) $\alpha_{q+2\sum_{i=1}^{k}M_{s+p_i}} > \dfrac{1}{M_{p_k}}$ $(k=1, \ldots, t)$,

(ii) $\sum_{i=1}^{t}\dfrac{1}{M_{p_i}}=1.$

Proof. Let s, q and $(\alpha_n,\ n\in\mathbf{N})$ be fixed. First we verify the following statement: For all $v, u\in\mathbf{N}$ there exists $r\in\mathbf{N}$ such that $\alpha_{v+2M_{u+r}}>\dfrac{1}{M_r}$. Suppose indirectly that $\alpha_{v+2M_{u+r}}\leqq\dfrac{1}{M_r}$ $(r\in\mathbf{N})$. Since $(\alpha_n,\ n\in\mathbf{N})$ is monotone, and $\limsup m<\infty$, therefore

$$\sum_{n=v+2M_u}^{\infty}\alpha_n^2=\sum_{r=0}^{\infty}\ \sum_{i=v+2M_{u+r}}^{v+2M_{u+r+1}}\alpha_i^2\leqq\sum_{r=0}^{\infty}2(M_{u+r+1}-M_{u+r})\frac{1}{M_r^2}<\infty,$$

but this is a contradiction, since $\sum_{n=1}^{\infty}\alpha_n^2=\infty$. Part (i) of Lemma 2 is a simple consequence of this statement.

In order to prove part (ii) let

$$p_1:=\min\left\{n\in\mathbf{N}|\alpha_{q+2M_{s+n}}>\frac{1}{M_n}\right\}$$

and

$$p_{k+1}:=\min\left\{n\in\mathbf{N}|\alpha_{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+n}}>\frac{1}{M_n},\ n\geqq p_k\right\}\quad (k\in\mathbf{N}).$$

The existence of p_k's $(k\in\mathbf{N})$ follows from the above statement. Since $(\alpha_n,\ n\in\mathbf{N})$ is monotone, therefore from the minimum property of p_k's and from $\limsup m<\infty$ we have

$$\sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}}^{q+2\sum_{i=1}^{k+1}M_{s+p_i}-1}\alpha_n^2=\sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}}^{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+p_k}-1}\alpha_n^2+\sum_{l=p_k}^{p_{k+1}-1}\ \sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+l}}^{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+l+1}-1}\alpha_n^2\leqq$$

$$\leqq 2M_{s+p_k}\frac{1}{M_{p_k-1}^2}+\sum_{l=p_k}^{p_{k+1}-1}2(M_{s+l+1}-M_{s+l})\frac{1}{M_l^2}<C_{s,m}\frac{1}{M_{p_k}}\quad (k\in\boldsymbol{P})$$

(where $C_{s,m}>0$ depends only on s and m), whence by $\sum_{n=1}^{\infty}\alpha_n^2=\infty$

$$\sum_{k=1}^{\infty}\frac{1}{M_{p_k}}=\infty.$$

On account of the monotonicity of $(p_k, k\in\mathbf{N})$ $M_{p_{k+1}}$ is divisible by M_{p_k} $(k\in\mathbf{N})$, namely there exists $l_k\in\mathbf{N}$ for which $\sum_{i=1}^{k}\frac{1}{M_{p_i}}=\frac{l_k}{M_{p_k}}$. If $l_k<M_{p_k}$ then $\sum_{i=1}^{k+1}\frac{1}{M_i}=$ $=\frac{l_k}{M_{p_k}}+\frac{1}{M_{p_{k+1}}}\leqq 1$. Thus the divergence of $\sum_{k=1}^{\infty}\frac{1}{M_{p_k}}$ implies the existence of $t\in\mathbf{N}$ such that $\sum_{k=1}^{t}\frac{1}{M_{p_k}}=1$. This completes the proof of Lemma 2.

4. Proof of Theorem

Let us fix a bounded Vilenkin system $\hat{G}_m$. Denote $(n_l,\ l\in\mathbf{N})$ a sequence of indices for which

$$\sum_{l=1}^{\infty}\frac{\Phi(2^{M_{n_l}})}{n_l}<\infty. \tag{2}$$

The existence of such a sequence follows from $\Phi(u)=o(\log\log u)$ $(u\to\infty)$. By means of $(n_l,\ l\in\mathbf{N})$ we define $(q_l,\ l\in\mathbf{N})$ inductively. Let $q_0:=0$. If q_l is given, then apply Lemma 2 for $q:=M_{q_l}$ and for $s:=M_{M_{n_l}+1}$. Thus we get the existence of $t_l\in\mathbf{N}$ and $p_{1,l}\leqq\ldots\leqq p_{t_l,l}$ $(p_{i,l}\in\mathbf{N},\ i=1,\ldots,t_l)$ for which

$$\alpha_{M_{q_l}+2\sum_{i=1}^{k}M_{M_{n_l}+1+p_{i,l}}}>\frac{1}{M_{p_{k,l}}}\quad (k=1,\ldots,t_l), \tag{3}$$

$$\sum_{k=1}^{t_l}\frac{1}{M_{p_{k,l}}}=1. \tag{4}$$

Let us see the polynomials (see Lemma 1) $Q_{n_l,p_{k,l},j_{k,l}}$ where $j_{k,l}:=M_{p_{k,l}}\sum_{i=1}^{k-1}\frac{1}{M_{p_{i,l}}}$ $(k=1,\ldots,t)$. Obviously

$$\bigcup_{k=1}^{t_l}I_{p_{k,l}}(e_{j_{k,l}})=G_m. \tag{5}$$

Define the numbers $s_{k,l}$ $(k\in\mathbf{P})$ by induction:

$$s_{1,l}:=\max\{M_{q_l},\ M_{M_{n_l}+1+p_{1,l}}\}$$

$$s_{k+1,l}:=\min\{n\in\mathbf{N}\,|\,M_{M_{n_l}+1+p_{k+1,l}}\ \text{is a divisor of } n,\quad n>\max\Delta(\psi_{s_{k,l}}Q_{n_l,p_{k,l},j_{k,l}})\}.$$

It is easy to see, that if M_t is a divisor of u $(t,u\in\mathbf{N})$ then $\psi_u\psi_v=\psi_{u+v}$ $(v<M_t)$. By reason of this the Vilenkin polynomial $F_{k,l}:=\psi_{s_{k,l}}Q_{n_l,p_{k,l},j_{k,l}}$ can be given by shift of the spectrum of $Q_{n_l,p_{k,l},j_{k,l}}$ $(k=1,\ldots,t_l)$. It is clear from the definition of $s_{k,l}$'s that

$$\min\Delta F_{1,l}>M_{q_l} \tag{6}$$

$$\Delta F_{k,l}\cap\Delta F_{h,l}=\emptyset\quad (k,h=1,\ldots,t_l,\ k\neq h). \tag{7}$$

Furthermore, since $\max \Delta F_{1,l} < M_{q_l} + 2M_{M_{n_l+1}+p_{1,l}}$ and $s_{k+1,l} - \max \Delta F_{k,l} \leqq \leqq M_{M_{n_l+1}+p_{k+1,l}}$ therefore we have by induction

(8) $$\max \Delta F_{k,l} < M_{q_l} + 2 \sum_{i=1}^{k} M_{M_{n_l+1}+p_{i,l}} \quad (k = 1, \ldots, t_l).$$

Obviously there exists $h_k \in \mathbf{N}$ such that

(9) $$\Delta F_{k,l} \subset [M_{h_k}, M_{h_k+1}) \quad (k = 1, \ldots, t_l).$$

$F_{k,l}$ preserves evidently the nice properties of $Q_{n_l, p_{k,l}, j_{k,l}}$, i.e.

(10) $$\|F_{k,l}\|_1 = 1, \quad \Omega F_{k,l}(x) > CM_{p_{k,l}} n_l, \quad \operatorname{supp} F_{k,l} \subset I_{p_{k,l}}(e_{j_{k,l}})$$
$$\big(x \in I_{p_{k,l}}(e_{j_{k,l}}), \quad k = 1, \ldots, t_l\big).$$

Let

$$f_l := \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} F_{k,l}.$$

If $s \in \Delta f_l$ then by (7) there exists a number k uniquely determined ($1 \leqq k \leqq t_l$) such that $s \in \Delta F_{k,l}$. Thus (3), (8) and (10) imply that

(11) $$|\hat{f}_l(s)| = \frac{1}{M_{p_{k,l}} n_l} \hat{F}_{k,l}(s) \leqq \frac{1}{M_{p_{k,l}} n_l} \|F_{k,l}\|_1 =$$
$$= \frac{1}{M_{p_{k,l}} n_l} < \alpha_s \quad (s \in \Delta F_{k,l}), \quad \text{viz.} \quad |\hat{f}_l(s)| < \alpha_s \quad (s \in \mathbf{N}).$$

By reason of (5) there exists for all $x \in G_m$ $1 \leqq k \leqq t_l$ such that $x \in I_{p_{k,l}}(e_{j_{k,l}})$ and then by (7), (10) we have

(12) $$\Omega f_l(x) \geqq \frac{1}{M_{p_{k,l}} n_l} \Omega F_{k,l}(x) > C.$$

On the other hand it follows from (9) that

$$f_l^* \leqq \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} F_{k,l}^* = \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} |F_{k,l}|.$$

The estimation

(13) $$\max |Q_n| \leqq 2^{M_{n_l}}$$

is trivial from the definition of Q_n (see [7] p. 362). Taking into consideration the construction of $F_{k,l}$'s and of f_l we have by (10), (13) that

$$\max_{x \in I_{p_{k,l}}(e_{j_{k,l}})} f_l^*(x) \leqq \frac{1}{M_{p_{k,l}} n_l} \max |F_{k,l}| \leqq 2^{M_{n_l}} \quad (k = 1, \ldots, t_l).$$

Thus

$$(f_l^* \Phi(f_l^*))(x) \leqq \frac{1}{M_{p_{k,l}} n_l} |F_{k,l}(x)| \Phi(2^{M_{n_l}}) \quad (x \in I_{p_{k,l}}(e_{j_{k,l}})), \quad (k = 1, \dots t_l),$$

consequently by reason of (4) and (10)

$$\|f_l^* \Phi(f_l^*)\|_1 \leqq \frac{\Phi(2^{M_{n_l}})}{n_l} \tag{14}$$

is valid. Now we define the sequence $(q_l,\ l \in \mathrm{N})$ inductively. Let $q_{l+1} := \min \{n \in \mathrm{N} | M_n > \max \Delta f_l\}$, and define f_{l+1} in the same manner as f_l $(l \in \mathrm{N})$. Denote f the following function

$$f := \sum_{l=0}^{\infty} f_l.$$

Since $\Delta f_k \cap \Delta f_j = \emptyset$ $(k, j \in \mathrm{N},\ k \neq j)$ and $\Delta f = \bigcup_{l=0}^{\infty} \Delta f_l$, therefore by (12) we have the everywhere divergence of the Vilenkin—Fourier series of f. (11) yields

$$|\hat{f}(s)| < \alpha_s \quad (s \in \mathrm{N}),$$

furthermore (2) and (14) furnish

$$\|f^* \Phi(f^*)\|_1 \leqq \sum_{l=0}^{\infty} \|f_l^* \Phi(f_l^*)\|_1 \leqq \sum_{l=0}^{\infty} \frac{\Phi(2^{M_{n_l}})}{n_l} < \infty,$$

i.e. $f \in H\Phi(H)$. This completes the proof of Theorem.

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